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Hardy's Inequality for Dirichlet Forms

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We prove an abstract form of Hardy's L^2 inequality, in which the Dirichlet integral is replaced by the Dirichlet form of a general symmetric Markov process. A number of examples are provided. © 2000 Academic Press

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1. INTRODUCTION

In the course of providing a new proof of Hilbert's double series theorem, G. H. Hardy [14] discovered the inequality

$$\int_0^\infty \frac{u(x)^2}{x^2}\,dx \le 4\int_0^\infty [u'(x)]^2\,dx, \tag{1.1}$$

which is valid for absolutely continuous u such that $u(0) = 0$ and $u' \in L^2(0,\infty)$. There is a vast literature concerning generalizations of (1.1); see [19] for a recent survey. Our aim in this note is to prove an abstract form of (1.1) in which the "Dirichlet integral" appearing on the right is replaced by the Dirichlet form of a general symmetric Markov process.

To illustrate the basic idea, let us sketch a "Brownian motion proof" of (1.1). The infinitesimal generator of Brownian motion on $]0,\infty[$ with an absorbing barrier at 0 is the second-order differential operator

$$Lf = \tfrac{1}{2}f'' \tag{1.2}$$

acting on smooth enough functions that vanish at 0. The associated Dirichlet form is given by

$$\mathscr{E}(f,g) := -\int_0^\infty Lf(x)g(x)\,dx = \tfrac{1}{2}\int_0^\infty f'(x)g'(x)\,dx \tag{1.3}$$

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provided f and g are smooth and of compact support. The function $w(x) := x^{1/2}$ satisfies $Lw(x) + w(x)/8x^2 = 0$ on $]0,\infty[$. In particular, $Lw < 0$ on $]0,\infty[$, so that w is a positive superharmonic function of the absorbed Brownian motion. The Doob w-transform of absorbed Brownian motion is the diffusion on $]0,\infty[$ with infinitesimal generator

$$L^w f(x) := L(wf)(x)/w(x) = \tfrac{1}{2} f''(x) + \frac{1}{2x} f'(x) - \frac{1}{8x^2} f(x). \quad (1.4)$$

It is easy to check that L^w is symmetric as an operator on $L^2(w^2\, dx)$ and that the associated Dirichlet form is given on the diagonal by

$$\begin{aligned} \mathscr{E}^w(f,f) &:= -\int_0^\infty L^w f(x) f(x) w(x)^2\, dx \\ &= \tfrac{1}{2}\int_0^\infty [f'(x)]^2 x\, dx + \tfrac{1}{8}\int_0^\infty f(x)^2 x^{-1}\, dx \\ &\geq \tfrac{1}{8}\int_0^\infty f(x)^2 x^{-1}\, dx, \end{aligned} \quad (1.5)$$

provided f is a smooth function of compact support. But for such f we also have $\mathscr{E}^w(f,f) = \mathscr{E}(wf, wf)$. Using this observation and defining $u = wf$, we deduce from (1.5) that

$$\mathscr{E}(u,u) \geq \tfrac{1}{8}\int_0^\infty [u(x)/w(x)]^2 x^{-1}\, dx = \tfrac{1}{8}\int_0^\infty u(x)^2 x^{-2}\, dx, \quad (1.6)$$

provided u is smooth and of compact support. Hardy's inequality (1.1) now follows for general u by approximation.

This method of proof for inequalities like (1.1) originates in the work of P. R. Beesack [2]. Our own interest in the subject was sparked by the paper [1] of A. Ancona, in which the validity of the Hardy inequality

$$\int_D \frac{u(x)^2}{\operatorname{dist}(x,\partial D)^2}\, dx \leq C\int_D |\nabla u(x)|^2\, dx, \quad (1.7)$$

for a Euclidean domain D, was shown to be equivalent to the existence of a positive superharmonic function w and a constant $\delta > 0$ such that

$$\Delta w + \delta w/\operatorname{dist}(\cdot,\partial D)^2 \leq 0 \quad (1.8)$$

in D. (The realization that the existence of an auxiliary function playing the role of w is a *necessary* consequence of an inequality like (1.7) is apparently due to Tomaselli [23].) Our aim is to show that such an equivalence holds quite generally.

We now describe our main result. Let $(\mathscr{E}, \mathscr{D})$ be the Dirichlet form associated with a transient symmetric strong Markov process X. Thus the transition semigroup of X is assumed to be self-adjoint with respect to some σ-finite measure m on the state space E of X. We use L to denote the L^2-infinitesimal generator of X and $U := -L^{-1}$ to denote the potential operator. A function $w\colon E \to [0,\infty]$ is *superharmonic* provided it is excessive (with respect to X) and finite m-a.e. A superharmonic function w admits a (unique) Riesz decomposition $w = U(\mu) + h$, where h (the harmonic part of w) is a superharmonic function that specifically dominates no nonzero potential. The (positive) measure μ can be uniquely decomposed as $\mu_0 + \mu_1$, where μ_0 charges no X-exceptional set and μ_1 is carried by an X-exceptional set. (A set $B \subset E$ is *X-exceptional* provided $P^m(X_t \in B \text{ for some } t > 0) = 0$. The term "quasi-everywhere" (q.e.) means "outside an X-exceptional set.") We call the measure μ_0 the *Riesz charge* of w and use the symbolism $\mu_0 = -\mathscr{L}w$ to indicate this relationship. For example, if $w = Uf$ is the potential of the function $f \geq 0$ then $-\mathscr{L}w = f \cdot m$.

(1.9) THEOREM. *Let ν be a σ-finite measure on E.*

(a) *Suppose there is a constant $\delta > 0$ and a strictly positive superharmonic function w such that*

$$\mathscr{L}w + (\delta w) \cdot \nu \leq 0. \tag{1.10}$$

Then

$$\int_E \tilde{u}^2 \, d\nu \leq \delta^{-1} \cdot \mathscr{E}(u,u) \qquad \forall u \in \mathscr{D}, \tag{1.11}$$

where $\tilde{u}$ is any quasi-continuous m-version of u.

(b) *Conversely, suppose there is a constant $0 < C < \infty$ such that*

$$\int_E u^2 \, d\nu \leq C \cdot \mathscr{E}(u,u), \tag{1.12}$$

whenever $u \in \mathscr{D}$ is the difference of bounded superharmonic elements of $\mathscr{D}$. Then for each $\delta \in]0, C^{-1}[$ there is a strictly positive superharmonic function $w \in \mathscr{D}$ such that (1.10) *holds.*

(1.13) *Remarks.* (a) The transience hypothesis imposed on X is entirely natural in the context of inequalities like (1.11). Indeed, as one easily deduces from [9, Sect. 1.5], a symmetric Markov process X is transient if and only if there exists a strictly positive function f on E such that $\int_E u^2 f \, dm \leq \mathscr{E}(u,u)$ for all $u \in \mathscr{D}$.

(b) As we shall see, (1.10) implies that ν is a *smooth measure* in the sense of [9, p. 80]. In particular, ν does not charge X-exceptional sets, so the left side of (1.11) is unambiguously defined because $\tilde{u}$ is uniquely determined modulo X-exceptional sets. If it is known *a priori* that $\nu U \ll m$, then μ charges no finely open X-exceptional set and the strict positivity of w can be weakened to the condition "$w > 0$ q.e.," the conclusion stated in (a) (and the smoothness of ν) still obtaining. On the other hand, if (1.12) holds whenever u is a difference of bounded superharmonic functions, then ν is necessarily smooth and (1.11) (with $\delta = C^{-1}$) holds for all $u \in \mathscr{D}$.

(c) Equality holds in (1.11) for a given $u \in \mathscr{D}$ if and only if u is a superharmonic element of $\mathscr{D}$ satisfying $\mathscr{L}u + (\delta u) \cdot \nu = 0$.

(d) For a different (and very general) potential theoretic approach to Hardy inequalities, see [20].

Let us illustrate Theorem (1.9) with two examples.

(1.14) EXAMPLE (Wirtinger's Inequality). If u: $[0, \pi] \to \mathbf{R}$ is absolutely continuous and periodic (in the sense that $u(0) = u(\pi)$) with $u' \in L^2([0, \pi])$, then

$$\int_0^\pi u(x)^2 \, dx \le \int_0^\pi [u'(x)]^2 \, dx \tag{1.15}$$

provided u vanishes somewhere in $[0, \pi]$. It is customary to state this inequality for periodic u with mean value $\pi^{-1} \int_0^\pi u(x) \, dx$ equal to 0. Of course such a u must vanish somewhere in $[0, \pi]$, and then by translation modulo π may be assumed to vanish at 0 and π. To prove (1.15) when $u(0) = u(\pi) = 0$, let X be Brownian motion on $E =]0, \pi[$ with absorbing boundaries. The function $w(x) := \sin x$ is then strictly positive and superharmonic, and

$$\mathscr{L}w(x) = \tfrac{1}{2} w''(x) \, dx = -\tfrac{1}{2} w(x) \, dx. \tag{1.16}$$

Thus (1.10) holds with $\delta = 1/2$ and $\nu(dx) = dx$, and (1.15) follows from Theorem (1.9)(a). Equality holds in (1.15) if and only if $u(x) = A \sin(x + B)$.

(1.17) EXAMPLE. Let μ be a smooth measure such that the potential $U(\mu)$ is finite a.e., hence superharmonic. Choose $f > 0$ with $Uf \le 1$. Then by Theorem (1.9), with $w = U(\mu) + \epsilon Uf$, $\nu = w^{-1} \cdot \mu$, and $\delta = 1$,

$$\int_E \frac{\tilde{u}^2}{U(\mu) + \epsilon Uf} \, d\mu \le \mathscr{E}(u, u) \qquad \forall u \in \mathscr{D}.$$

Using Fatou's lemma to pass to the limit as $\epsilon \to 0$, we obtain

$$\int_E \frac{\tilde{u}^2}{U(\mu)} \, d\mu \le \mathscr{E}(u, u) \qquad \forall u \in \mathscr{D}, \tag{1.18}$$

where the ratio on the left is taken to be 0 in indeterminate cases. In particular, if $C := \|U\mu\|_{L^\infty(m)} < \infty$, then

$$\int_E \tilde{u}^2 \, d\mu \le C\mathscr{E}(u,u) \qquad \forall u \in \mathscr{D}. \tag{1.19}$$

The inequality (1.19) has been established in the present context by P. Stollmann and J. Voigt [22, Theorem 3.1], using a method going back to T. Kato [16, Lemma VI.4.8a]. Z. Vondraček [24], extending work of Glover *et al*. [12], has proved (1.19) (with $4C$ instead of C) by adapting a method of K. Hansson [13]. See also [8, Theorem (4.24)], where (1.19) (with $2C$ instead of C) is obtained in a broader setting by a different method. Related matters are discussed at the beginning of Section 4.

In Section 2 we describe the precise context in which Theorem (1.9) will be proved. The proof of (1.9) is contained in Section 3, while Section 4 is devoted to further examples.

2. PRELIMINARIES

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a transient right Markov process as discussed in [11, 21]. We assume that the state space $(E, \mathscr{B}(E))$ of X is homeomorphic to a Borel subset of some compact metric space; here $\mathscr{B}(E)$ is the class of Borel subsets of E. Furthermore, we assume that the transition semigroup of X, defined by

$$P_t f(x) := P^x(f(X_t)), \qquad t \ge 0,\ f \in b\mathscr{B}(E), \tag{2.1}$$

maps $b\mathscr{B}(E)$ (the class of bounded real-valued $\mathscr{B}(E)$-measurable functions on E) into itself. In view of the symmetry hypothesis discussed below, this assumption of Borel measurability entails no loss of generality; see [8, Sect. 3]. By *transient* we mean that the potential kernel $U := \int_0^\infty P_t \, dt$ is proper, in the sense that there is a strictly positive $\mathscr{B}(E)$-measurable function q such that Uq is bounded.

To allow for the possibility $P_t 1_E(x) < 1$, a cemetery state Δ is adjoined to E as an isolated point, and the process is sent to Δ at its lifetime ζ. By convention any function (resp. measure) defined on E (resp. $\mathscr{E}^*$) is extended to the cemetery state Δ by declaring its value at Δ (resp. $\{\Delta\}$) to be 0.

Our final basic hypothesis is that of *symmetry*: We assume that there is a σ-finite measure m on $(E, \mathscr{B}(E))$ such that

$$(f, P_t g)_m = (P_t f, g)_m \qquad \forall f, g \in p\mathscr{B}(E), \tag{2.2}$$

where $(u, v)_m := \int_E uv\, dm$. In this case (P_t) restricted to $b\mathscr{B}(E) \cap L^2(m)$ extends uniquely to a strongly continuous contraction semigroup in $L^2(m)$. Of course, (2.2) implies that this semigroup is self-adjoint. The *Dirichlet form* associated with the symmetric process X is the bilinear form

$$\mathscr{E}(u, v) := \lim_{t \downarrow 0} t^{-1}(u, v - P_t v)_m \tag{2.3}$$

defined on the vector space

$$\mathscr{D} := \left\{ u \in L^2(m) \colon \sup_{t > 0} t^{-1}(u, u - P_t u)_m < \infty \right\}. \tag{2.4}$$

An alternative description of $(\mathscr{E}, \mathscr{D})$ is provided by the L^2-infinitesimal generator $(L, D(L))$ of the semigroup (P_t). Namely,

$$\mathscr{D} = D\left(\sqrt{-L}\right) \qquad \text{and} \qquad \mathscr{E}(u, v) = \left(\sqrt{-L}\,u, \sqrt{-L}\,v\right)_m. \tag{2.5}$$

The exit time $\inf\{t > 0\colon X_t \notin B\}$ of a Borel subset of E is denoted $\tau(B)$. An increasing sequence (B_n) of Borel subsets of E is a *nest* provided $P^m(\lim_n \tau(B_n) < \zeta) = 0$. The reader is referred to [17, Lemma IV.4.5] for a characterization of this notion in terms of the Dirichlet form $(\mathscr{E}, \mathscr{D})$.

Each element $u \in \mathscr{D}$ admits an m-modification $\tilde{u}$ such that $t \mapsto \tilde{u}(X_t)$ is right continuous on $[0, \infty[$ with left limits on $]0, \infty[$, P^m-a.s. The function $\tilde{u}$ is *quasi-continuous* in the sense that there is a nest (K_n) of compact subsets of E such that $\tilde{u}|_{K_n} \in C(K_n)$ for every n.

A Borel set $N \subset E$ is said to be *X-exceptional* provided $P^m(\tau(E \setminus N) < \zeta) = 0$. It can be shown that $N \in \mathscr{B}(E)$ is X-exceptional if and only if there is a nest of compacts (K_n) such that $N \subset \bigcap_n K_n^c$.

A measure ν on $(E, \mathscr{B}(E))$ is *smooth* provided (i) ν charges no X-exceptional element of $\mathscr{B}(E)$ and (ii) there is a nest of compacts (K_n) such that $\nu(K_n) < \infty$ for all n.

We shall make use of the *Revuz correspondence* between the class of smooth measures and the class of *positive continuous additive functionals* (PCAFs) of X. If $A = (A_t)$ is a PCAF of X, then the formula

$$\nu_A(f) := \uparrow \lim_{t \downarrow 0} t^{-1} P^m \int_0^t f(X_s)\, dA_s \tag{2.6}$$

defines a smooth measure ν_A. Conversely, given a smooth measure ν, there is an essentially unique PCAF A such that $\nu = \nu_A$. See [9, Sect. 5.1].

A universally measurable function $f\colon E \to [0, \infty]$ is excessive provided $t \mapsto P_t f(x)$ is decreasing and right-continuous on $[0, \infty[$ for every $x \in E$. If, in addition, $\{f = \infty\}$ is m-null (equivalently, X-exceptional) then we call f *superharmonic*. Because X is symmetric, every superharmonic function is quasi-continuous.

Let μ be a measure on $(E, \mathscr{B}(E))$ such that the potential μU is σ-finite and absolutely continuous with respect to m. The Radon–Nikodym derivative $d(\mu U)/dm$ can then be taken to be superharmonic [10, (3.6)]; we use the symbol $U(\mu)$ to denote this density. The absolute continuity of μU implies that μ charges no finely open X-exceptional set; for this reason $U(\mu)$ is uniquely determined modulo an X-exceptional set.

Each superharmonic function w admits a unique decomposition $w = U(\mu) + h$, where μ is as in the preceding paragraph and h is a superharmonic function that specifically dominates no nonzero potential $U(\nu)$. Decomposing μ as the sum $\mu_0 + \mu_1$ of a measure charging no X-exceptional set and a measure carried by an X-exceptional set, we obtain the *Riesz charge* μ_0, which will sometimes be denoted $-\mathscr{L}w$. If the excessive function $U(\mu)$ happens to be an element of $\mathscr{D}$, then $\mu_1 = 0$ and

$$\mathscr{E}(U(\mu), v) = \mu(\tilde{v}), \qquad \forall v \in \mathscr{D}.$$

Any superharmonic element of $\mathscr{D}$ is necessarily of the form $U(\mu)$, where μ is a smooth measure; see [5, Theorem (5.9)].

3. PROOF OF THEOREM (1.9)

(a) As noted in Remark (1.13)(a), ν is a smooth measure. Indeed, (1.10) means that

$$(\delta w) \cdot \nu \leq \mu_0, \tag{3.1}$$

where μ_0 is the Riesz charge of w. Because μ_0 is smooth and w is strictly positive and excessive (hence quasi-continuous), ν is smooth. Thus $\delta \cdot \nu$ is the Revuz measure of a unique PCAF A of X. Likewise, the Riesz charge μ_0 of w is smooth, hence the Revuz measure of a PCAF B. The inequality (3.1) means that the PCAF $\int_0^t w(X_s)\, dA_s$ is strongly dominated by B, in the sense that the difference $B_t - \int_0^t w(X_s)\, dA_s$ is itself a PCAF. Using this fact and Itô's formula, one can verify that the process $t \mapsto \exp(A_t) w(X_t)$ is a supermartingale, strictly positive on $[0, \zeta[$. Consequently, the semigroup

$$Q_t f(x) := P^x(\exp(A_t)(wf)(Y_t))/w(x), \qquad t \geq 0, \tag{3.2}$$

is well defined and operates as a strongly continuous sub-Markovian self-adjoint semigroup in $L^2(w^2 \cdot m)$; see [7, Sect. 4]. Let Y denote the $w^2 \cdot m$-symmetric right process associated with (Q_t) and let $(\mathscr{E}', \mathscr{D}')$ be its (quasi-regular) Dirichlet form. Because the law of Y is locally absolutely continuous with respect to that of X, the process $\exp(-A_t)$ can be viewed as a (continuous, decreasing) multiplicative functional of Y. If we now "kill" Y using $\exp(-A_t)$, then the resulting process is just the w-transform of X. (Recall that w is strictly positive and excessive.) It is easy to check that the Dirichlet form associated with the w-transform of X is given by

$$\mathscr{E}^w(u,v) = \mathscr{E}(wu,wv) \tag{3.3}$$

on the domain

$$\mathscr{D}^w := \{f\colon wf \in \mathscr{D}\}. \tag{3.4}$$

Putting this together with [6, Proposition 3], which describes the effect on a Dirichlet form of killing via a multiplicative functional such as $\exp(-A_t)$, we find that

$$\mathscr{E}'(u,v) + \delta\cdot(w\tilde{u},w\tilde{v})_\nu = \mathscr{E}(wu,wv) \tag{3.5}$$

for all $u,v \in \mathscr{D}^w$, part of the assertion being that $\mathscr{D}^w = \mathscr{D}' \cap L^2(\nu)$. Since $\mathscr{E}'$ is a Dirichlet form, it is positive semi-definite; thus (3.5) implies

$$\mathscr{E}(wu,wu) \geq \delta\cdot(w\tilde{u},w\tilde{u})_\nu \qquad \forall u \in \mathscr{D}^w, \tag{3.6}$$

which implies (1.11).

(b) The smoothness of ν can be deduced from (1.12) exactly as in the proof of [8, Theorem (4.24)] and then a standard approximation argument shows that the inequality in (1.11) holds for all $u \in \mathscr{D}$. We now follow the argument of Ancona [1]. Let us fix $\delta \in]0, C^{-1}[$ and consider the bilinear form

$$\mathscr{E}_\delta(u,v) := \mathscr{E}(u,v) - \delta\cdot(\tilde{u},\tilde{v})_\nu, \qquad u,v \in \mathscr{D}. \tag{3.7}$$

Clearly (1.11) implies that $\mathscr{E}_\delta$ is continuous on $\mathscr{D}\times\mathscr{D}$, and $\mathscr{E}_\delta$ is coercive:

$$\mathscr{E}_\delta(u,u) \geq (1-\delta C)\mathscr{E}(u,u), \qquad \forall u \in \mathscr{D}. \tag{3.8}$$

Since X is transient we can choose a strictly positive $f \in L^1(m)$ such that $Uf \leq 1$. By part (a) of (1.9) (applied with $w = Uf$ and $\delta = 1$) we have

$$\int_E u^2 f\,dm \leq \mathscr{E}(u,u). \tag{3.9}$$

Using (3.9) and (1.11) we see that if $u \in \mathscr{D}$, then

$$\left|\int_E uf\,dm\right| \leq \left[\int_E f\,dm\right]^{1/2}\left[\int_E u^2 f\,dm\right]^{1/2} \leq C_1\mathscr{E}(u,u)^{1/2}. \tag{3.10}$$

Thus the linear functional

$$u \mapsto \int_E uf\,dm, \qquad u \in \mathscr{D}, \tag{3.11}$$

is well-defined and continuous on $\mathscr{D}$. Since $\mathscr{E}_\delta$ is coercive, the Lax–Milgram theorem allows us to conclude that there exists a unique $w \in \mathscr{D}$ such that

$$\int_E \tilde{u} f\, dm = \mathscr{E}_\delta(w,u) \qquad \forall u \in \mathscr{D}. \tag{3.12}$$

Taking $u = w^-$ in (3.12) ($w^{\pm} := \max(\pm w, 0)$) and using the fact that $\mathscr{E}(w^+, w^-) \le 0$ (see [17, I.4.4(ii), I.4.17]), we find that

$$\int_E \tilde{w}^- f\, dm = \mathscr{E}(w, w^-) + \delta \int_E [\tilde{w}^-]^2\, d\nu \le -\mathscr{E}_\delta(w^-, w^-) \le 0,$$

which means that $\tilde{w}^- = 0$ q.e. From this and (3.12) we deduce that $\mathscr{E}(w,u) \ge 0$ for all $u \in p\mathscr{D}$; thus w can be taken to be excessive, and therefore quasi-continuous. Let μ denote the Riesz charge of w, so that

$$\mathscr{E}(w,u) = \mu(\tilde{u}) \qquad \forall u \in \mathscr{D}. \tag{3.13}$$

Using (3.13) to rewrite (3.12), we obtain

$$\int_E \tilde{u}\, d\mu = \int_E \tilde{u}(f\, dm + \delta w\, d\nu) \qquad \forall u \in \mathscr{D}. \tag{3.14}$$

Thus,

$$\delta w \cdot \nu \le f \cdot m + \delta w \cdot \nu = \mu, \tag{3.15}$$

which is (1.10).

Now $w \ge 0$ everywhere and the set $B := \{w = 0\}$ is finely closed and absorbing. Moreover, (3.15) implies that $w = U(\mu) \ge U(1_B \mu) = U(1_B f)$ q.e., and so $U(1_B f) = 0$ q.e. on B. The domination principle now implies that $U(1_B f) = 0$ q.e., which forces $m(B) = 0$ since $f > 0$. Since B is absorbing, it is necessarily X-exceptional. The excessive function $w^* := w + P_B U f$ is a strictly positive element of $\mathscr{D}$, is equal to w q.e., and $\mathscr{L} w^* = \mathscr{L} w$. Replacing w by w^* finishes the proof.

(3.16) *Remark.* Given $B \in \mathscr{B}(E)$ and its hitting time $T_B := \inf\{t > 0\colon X_t \in B\}$, define the *equilibrium potential* $\varphi_B(x) := P^x(T_B < \infty)$. Evidently φ_B is a bounded excessive function, and if $\varphi_B \in \mathscr{D}$, then $\varphi_B = U(\pi_B)$ q.e. for some measure π_B, called the equilibrium measure of B. For such B we define the (fine 0-order) capacity Cap by

$$\operatorname{Cap}(B) := \pi_B(E) = \pi_B(\tilde{B}) = \mathscr{E}(\varphi_B, \varphi_B), \tag{3.17}$$

where $\tilde{B}$ denotes the fine closure of B. (If $\varphi_B \notin \mathscr{D}$ then we set $\operatorname{Cap}(B) = \infty$.) It is easy to see that $\varphi_B = 1$ q.e. on $\tilde{B}$; thus $1_B \le \varphi_B^2$ q.e. Consequently, if (1.12) holds then

$$\nu(B) \leq C \cdot \mathrm{Cap}(B), \qquad \forall B \in \mathscr{B}_E. \tag{3.18}$$

Conversely, suppose that (3.18) holds. Then the estimate [13, Theorem 1.6; 24, Proposition 2]

$$\int_0^\infty \mathrm{Cap}(|\tilde{u}| > t) 2t\,dt \leq 4\mathscr{E}(u,u), \qquad \forall u \in \mathscr{D}, \tag{3.19}$$

implies that

$$\int_E \tilde{u}^2\,d\nu = \int_0^\infty \nu(|\tilde{u}| > t) 2t\,dt \leq 4C\mathscr{E}(u,u), \qquad \forall u \in \mathscr{D}. \tag{3.20}$$

In short, modulo determination of sharp constants, a capacity estimate of the form (3.18) is equivalent to the Hardy inequality (1.12). This observation is a special case of a broad theory of such equivalences (for p-Laplacians in Euclidean space) due to V. G. Maz'ja [18, Chap. 2].

4. EXAMPLES

For the sake of definiteness we confine our attention to Brownian motion and closely related symmetric diffusions. However, all of the examples discussed below have their analogues in the broader context of symmetric Markov diffusion processes.

Recall that if E is a domain in Euclidean space $\mathbf{R}^n$, then $H_0^1(E)$ is the closure of $C_0^\infty(E)$ (the smooth real-valued functions with support in a compact subset of E) with respect to the norm $[\int_E |\nabla u|^2 + u^2\,dx]^{1/2}$. Here m is the Lebesgue measure on E. Each $u \in H_0^1(E)$ is an element of $L^2(m)$ whose distribution-sense gradient ∇u is also an element of $L^2(m)$. The bilinear form

$$\mathscr{E}(u,v) := \tfrac{1}{2}\int_E \nabla u \cdot \nabla v\,dm, \qquad u,v \in H_0^1(E), \tag{4.1}$$

is then the Dirichlet form of standard Brownian motion killed upon first exiting E. This process is transient if $n \geq 3$ or if $n \leq 2$ and $\mathbf{R}^2 \setminus E$ is non-polar, and we assume this to be the case in the first four examples below.

(4.2) EXAMPLE (cf. [3, Proposition 1.1]). Let $\psi \geq 0$ be locally an element of $H_0^1(E)$; that is, $\varphi\psi \in H_0^1(E)$ whenever $\varphi \in C_0^\infty(E)$. Then ψ admits a quasi-continuous m-version $\tilde{\psi} \geq 0$, which we assume to be strictly positive q.e. Suppose furthermore that $|\nabla\psi| \le 1$ m-a.e., and that $\mathscr{L}\psi \le C\psi^{-1}\,dx$ for some $C > 1/2$. (For instance, $\psi(x) = \operatorname{dist}(x, x_0)$, where $x_0 \in E$ is fixed, provided $n \ge 3$, in which case $C = (n-1)/2$.) If $\beta := C - 1/2 > 0$, then $w := \psi^{-\beta}$ is superharmonic; in fact

$$\begin{aligned}\mathscr{L}w &= -\beta\psi^{-\beta-1}\mathscr{L}\psi + \frac{\beta(\beta+1)}{2}\psi^{-\beta-2}|\nabla\psi|^2\,dx\\ &\le \psi^{-\beta-2}\left[-\beta C + \frac{\beta(\beta+1)}{2}\right]dx,\end{aligned}$$

so that (1.10) holds with $\delta := (C-1/2)^2/2$ and $\nu(dx) = \psi(x)^{-2}\,dx$. Consequently,

$$\int_E \frac{u^2}{\psi^2}\,dx \le (C-1/2)^{-2}\int_E |\nabla u|^2\,dx, \qquad \forall u \in H_0^1(E). \tag{4.3}$$

(4.4) EXAMPLE. Suppose that $h > 0$ is harmonic on E and that $|\nabla h|^2 \ge Ch^\alpha$ for some real α and $C > 0$. Defining $w := \sqrt{h}$ we have

$$Lw = -\frac{1}{8}wh^{-2}|\nabla h|^2 \le -\frac{C}{8}wh^{\alpha-2},$$

so we can choose $\nu(dx) = h(x)^{\alpha-2}\,dx$ and $\delta = C/8$ in Theorem (1.9)(a), with the result that

$$\int_E \tilde{u}^2 h^{\alpha-2}\,dx \le \frac{4}{C}\int_E |\nabla u|^2\,dx, \qquad \forall u \in H_0^1(E). \tag{4.5}$$

(4.6) EXAMPLE. Assume now that E is convex (and not all of $\mathbf{R}^n$), and set $\rho(x) := \operatorname{dist}(x, E^c)$. Then ρ is concave (hence superharmonic) and strictly positive in E. We take $w = \sqrt{\rho}$, $\delta = 1/8$, and $\nu(dx) = [\rho(x)]^{-2}\,dx$, thereby obtaining the classical Hardy inequality

$$\int_E \frac{u^2}{\rho^2}\,dx \le 4\int_E |\nabla u|^2\,dx, \qquad \forall u \in H_0^1(E). \tag{4.7}$$

(4.8) EXAMPLE (Chernoff's Inequality [4]). In this final example we take X to be the 1-dimensional Ornstein–Uhlenbeck process, with generator

$$Lu(x) = \frac{1}{2}\left[u''(x) - xu'(x)\right]. \tag{4.9}$$

Then X is symmetric with respect to the standard normal distribution

$$m(dx) = [2\pi]^{-1/2}\exp(-x^2/2)\,dx, \tag{4.10}$$

and the associated Dirichlet form is given by

$$\mathscr{E}(u,v) = \tfrac{1}{2}\int_{\mathbf{R}} u'(x)\cdot v'(x)m(dx) \tag{4.11}$$

on the domain $\mathscr{D}$ consisting of those functions $u \in L^2(m)$ that are absolutely continuous, with derivative in $L^2(m)$. Now X is not transient, but we can apply Theorem (1.9) to X killed on first hitting $\{0\}$; the resulting Dirichlet form is just (4.11) restricted to those elements of $\mathscr{D}$ that vanish at 0. Taking $w(x) := |x|$, $\delta := 1/2$, and $\nu(dx) := m(dx)$, we obtain

$$\int_{\mathbf{R}} u(x)^2 m(dx) \le \int_{\mathbf{R}} [u'(x)]^2 m(dx), \tag{4.12}$$

provided u is absolutely continuous with $u' \in L^2(m)$ and $u(0) = 0$. Because $\int_{\mathbf{R}}[u(x) - m(u)]^2 m(dx) \le \int_{\mathbf{R}}[u(x) - u(0)]^2 m(dx)$, (4.12) implies "Chernoff's inequality"

$$\int_{\mathbf{R}} [u(x) - m(u)]^2 m(dx) \le \int_{\mathbf{R}} [u'(x)]^2 m(dx), \tag{4.13}$$

for all absolutely continuous u with derivative $u' \in L^2(m)$.

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